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Chvatal–Gomory–tier cuts for general integer programs

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Abstract

In this paper, we introduce a new class of cutting planes called *Chvatal–Gomory* (*CG*)*-tier cuts*. These cuts are predicated on a scaling parameter d and an integer tier-level parameter p, where $1 \leqslant p \leqslant d$. The cut generation process begins by scaling a given source row by d and deriving a standard Chvatal–Gomory (CG) cut at level one, but then repeats this process a total of p times, each time using the previous cut as a source row along with a scaling parameter value that is successively decremented by one. We derive a closed-form expression for this cut depending on the parameters p and d, so that the resulting CG-tier cut can be composed directly without actually implementing the foregoing recursive process. Furthermore, we study the variational properties of the cut coefficients as a function of the parameters d and p in order to facilitate a prescription of these parameter values for constructing strong CG-tier cuts that tighten the representation along specified desired dimensions. Several examples are provided to offer insights into the strength and nature of these cuts, including an illustration that these cuts can produce strong valid inequalities that are not obtainable via rank-one CG cuts. This paper focuses on the underlying derivation and structure of this class of CG-tier cuts; a follow-on study will address related implementation and computational issues.

Keywords: Integer programming; Gomory cuts; Chvatal–Gomory (CG) cuts; CG-tier cuts.

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1. Introduction

In this paper, we introduce a new class of *Chvatal–Gomory-tier cuts* for pure general (not necessarily binary) integer programs. These cuts begin by deriving a standard Chvatal–Gomory (CG) cut (see [7,15–19]) from a suitable source row by using a scaling parameter *d*, and then repeatedly applying the same cut generation process successively to each resulting cut some *p* times (including the first application), while decrementing the scaling parameter by one each time, where $1 \leqslant p \leqslant d$. This recursive process produces a *tier cut*, or a *Chvatal–Gomory* (*CG*)-*tier* cut, which depends on the parameters *p* and *d*. Although an inordinate number of possible cutting planes can be generated by varying the *scaling parameter d* and the *tier-level p* in this process, we show that it is possible to identify the strongest alternatives for selecting *d* for each tier-level choice, depending on specified cut objectives. Thus, instead of requiring the application of an iterative algorithm that searches over each tier for a strongest cut, and in turn searches over the resultant cuts for the best one to use as the source equation for the next tier, we specify a single cut generating mechanism that automatically yields the strongest CG-tier cut for any tier-level parameter choice, in a defined sense. Moreover, we can also specify the tier-level *p* to use for generating the strongest CG-tier cut subject to various constraining relationships imposed on the cut coefficients. This feature is useful from the practical goal of constructing "tailored cuts" that are deep in chosen dimensions. Coupled with the fact that the CG-tier cuts can be directly composed upon specifying the values of their parameters, the resulting cutting planes combine strength with speed of generation. In particular, as we shall demonstrate in this paper, a variety of strong cuts can be derived in this fashion, including cut instances that cannot be generated as level-one or rank-one CG cuts.

After nearly three decades of dormancy, Gomory (or CG) cuts have been receiving renewed interest by researchers, mainly due to the significantly enhanced modern-day computational capabilities coupled with improved techniques for selectively generating and implementing such cuts within a branch-and-cut framework. Balas et al. [1] and Ceria [5], for example, discuss the generation of rounds of lifted CG cuts that are valid for all nodes of the enumeration tree within such a framework for 0–1 mixed-integer programs. Commercial softwares such as CPLEX, OSL, and XPRESS-MP have all embraced CG cuts in their arsenal of cutting planes implemented within branch-and-bound/cut routines. For example, Savelsbergh [25] discusses the value of CG cuts in solving some challenging instances of integer programs using XPRESS-MP.

Various other improved cut generation strategies have also been proposed and demonstrated to yield favorable computational results. Ceria et al. [6] present a strategy for combining and strengthening CG cuts, Caprara et al. [4] study the seperation of maximally violated mod-k type of CG cuts, and Fischetti [3] and Fischetti et al. [10] explore and experiment with the generation of CG cuts for general integer programs using restricted weights of 0 or $1/2$ in the surrogation process. Gunluk and Pochet [20] describe a mixing procedure for deriving valid inequalities for mixed-integer programs that involves generating some 10 base inequalities from the source row after multiplying it by $\pm k$, for $k = 1, \ldots, 5$, and then applying a special strengthening technique. In a similar vein, Cornuejols et al. [8] study the generation of *k-cuts*, which are Gomory mixed-integer cuts (or strengthened fractional cuts) that are derived after first multiplying the source row by a positive integer *k*.

Their computational results reveal that such cuts are effective for pure integer knapsack problems, but are not so attractive in the presence of either multiple constraints or continuous variables.

In the context of generating CG cuts for 0–1 programs based on knapsack polytopes with or without generalized upper bounding constraints, Glover et al. [14] show how a suitable surrogate analysis can be performed while solving a separation problem to derive strong CG cuts. Marchand and Wolsey [22] also discuss an effective technique for solving the separation problem of generating a mixed-integer rounding (MIR) or CG cut to delete a given solution by using a suitable aggregation of the original constraints, accompanied by a complementation and scaling process. Many of these contributions can be viewed in the light of the earlier work on surrogate analysis due to Glover [11,12], which laid the foundation for rapidly generating cuts that dominate the original fractional cuts of Gomory [17]. It is worth noting here that whereas Eisenbrand [9] has proven that, in general, the separation problem of finding a CG cut to delete a given solution within a polyhedron, if one exists, is strongly NP-Hard, this problem can indeed be solved in polynomial time when the given solution belongs to a face of the underlying polyhedron, as shown by Letchford [21]. Many other generalized classes of cuts that relate to or subsume CG cuts have also been presented in the literature including superadditive cuts as discussed by Nemhauser and Wolsey [23], mixed-integer rounding cuts as described by Marchand and Wolsey [22], and Foundation-Penalty cuts as proposed by Glover and Sherali [13].

The remainder of this paper is organized as follows. In the next section, we introduce the class of CG-tier cuts predicated on the aforementioned scaling parameter d and integer tier-level parameter p, where $1 \leqslant p \leqslant d$, and we derive a closed-form expression for the cut coefficients generated by the underlying recursive technique. Thereafter, in Section 3, we explore the variation in these cut coefficients as a function of $d \geqslant p$ for a fixed value of $p \geqslant 1$, as also the effect of varying the tier-level p itself in concert with the scaling parameter d. This provides a facility to prescribe values of these parameters, depending on the objective of strengthening certain specified cut coefficients. The quality of the resulting cuts in contrast with the rank-one CG cut, and the ability to derive strong cuts that focus on tightening the representation along specified dimensions, is illustrated via several examples in Section 4. In particular, using the example afforded by Ben-Israel and Charnes [2], we demonstrate that valid inequalities that cannot be possibly derived using CG cuts can indeed be recovered by CG-tier cuts. Finally, we close in Section 5 with a summary and offer extensions and recommendations for future work, which would include a detailed computational study on generating and implementing suitable members from this class of CG-tier cuts.

2. Derivation of Chvatal–Gomory (CG)-tier cuts

We take as our starting point the equation (*source row*)

$$\sum_{i=1}^{n} a_i x_i = a_0. \tag{1}$$

The x_i are assumed to be nonnegative integer variables, and we retain this assumption throughout the paper. For a suitable $\lambda \neq 0$, the Chvatal–Gomory “all-integer” cut for this equation is given by

$$s + \sum_{i=1}^{n} \lfloor a_i/\lambda \rfloor x_i = \lfloor a_0/\lambda \rfloor \tag{2}$$

with s being a nonnegative integer. By setting $d = -\lambda$, and observing that $\lfloor -a \rfloor = -\lceil a \rceil$ for any number a, we obtain the related cut equation

$$z - \sum_{i=1}^{n} \lceil a_i/d \rceil x_i = -\lceil a_0/d \rceil, \tag{3}$$

where, although $z \equiv s$, if $d = -\lambda$, we have used a different (nonnegative, integer-valued) slack variable z in (3) to emphasize a general use.

Finally, by dividing Eq. (1) through by a nonzero parameter q and adding the result to (3), we obtain the equation

$$z + \sum_{i=1}^{n} (a_i/q - \lceil a_i/d \rceil) x_i = a_0/q - \lceil a_0/d \rceil. \tag{4}$$

The foregoing is a rather convenient cut representation insofar as it directly yields all of the cuts associated both with Gomory's Method of Integer Forms and his All-Integer Algorithm by varying the values of the parameters q and d. Thus, when $q = 8$ and $d < 0$, one obtains the cuts associated with the All Integer Algorithm and when $1/q = 1/d =$ an integer, one obtains the finite cyclic set of cuts associated with the Method of Integer Forms. Moreover, Glover [11] has shown that values for q and d other than those that define the usual Gomory cuts can be specified to yield stronger cuts than the ones obtained by the Method of Integer Forms, while at the same time sharing certain structural properties in common with these cuts.

Eq. (4) provides the basis for the cuts to be developed in this paper under the assumption that $q = 1$. Thus, we will be specifically concerned (at level one) with the cutting plane

$$z + \sum_{i=1}^{n} (a_i - \lceil a_i/d \rceil) x_i = a_0 - \lceil a_0/d \rceil. \tag{5}$$

Now, consider assigning the role of (1)–(5), which is to be the new source row for a second cut of the form (5). Suppose also that this second cut, which is based upon (5) as its source row, is to be generated by using a value of d that is one less than the value that initially produced (5) from (1). The new cut may thereby be written as follows:

$$\begin{aligned} &z' + (1 - \lceil 1/(d-1) \rceil) z + \sum_{i=1}^{n} (a_i - \lceil a_i/d \rceil - \lceil (a_i - \lceil a_i/d \rceil)/(d-1) \rceil) x_i \\ &\quad = a_0 - \lceil a_0/d \rceil - \lceil (a_0 - \lceil a_0/d \rceil)/(d-1) \rceil. \end{aligned} \tag{6}$$

As long as $d - 1 \geqslant 1$, the coefficient of z in this new equation will be zero, leaving only the single new (integer) variable z' from which the "prime" superscript may then be dropped.

We will always select the value of d so that the intermediate z-variable will be eliminated in this fashion.

While Eq. (6) is not a particularly convenient cut representation, the expression that results by repeating the procedure that gives (6) from (5) several more times is, of course, far more cryptic. As an initial step toward simplification, we will give a precise recursive definition of the cut equation obtained by applying the foregoing procedure p times (counting the initial derivation of (5)).

Given a number d and an integer p such that $1 \leqslant p \leqslant d$, and given a coefficient a (with the subscript i deleted for notational convenience), define recursively

$$a^0 = a, \text{ and with } d^k \equiv d - k, \text{ for } k = 0, 1, \ldots, p-1, \text{ let}$$

$$a^k = a^{k-1} - \left\lceil \frac{a^{k-1}}{d^{k-1}} \right\rceil, \text{ for } k = 1, \ldots, p.$$

Then the equation

$$z + \sum_{i=1}^{n} a_i^p x_i = a_0^p \tag{7}$$

with z a nonnegative integer variable, defines what we shall call a *CG-tier cut*, and may be seen by our earlier remarks to follow from the source row (1) by successively applying for p steps the transformation that gives (5) from (1), decrementing the value of d by one at each step. (The intermediate z variables created by this process all vanish since the decremented value of d is always greater than or equal to 1 because $1 \leqslant p \leqslant d$ implies that $d-(p-1) \geqslant 1$.)

We first derive below a nonrecursive representation of (7). Toward this end, for any integer $k \geqslant 0$, define

$$\beta^k \equiv \lceil a^k/d^k \rceil d^k - a^k$$

and

$$\gamma^k \equiv \beta^k - \lfloor \beta^k/d^{k+1} \rfloor d^{k+1}.$$

It may readily be verified that the above definitions imply $d^k > \beta^k \geqslant 0$ and $d^{k+1} > \gamma^k \geqslant 0$. Moreover, we have the following results holding true.

Lemma 1. $\lceil a^{k+1}/d^{k+1} \rceil = \lceil a^k/d^k \rceil - \lfloor \beta^k/d^{k+1} \rfloor$.

Proof. From the definition of β^k and a^{k+1}, and noting that $d^{k+1} = d^k - 1$, we obtain

$$(a^{k+1} + \beta^k)/d^{k+1} = \lceil a^k/d^k \rceil.$$

Also, from the definition of γ^k, we have

$$\beta^k/d^{k+1} = \lfloor \beta^k/d^{k+1} \rfloor + \gamma^k/d^{k+1}.$$

Combining and rearranging, we get

$$a^{k+1}/d^{k+1} = \lceil a^k/d^k \rceil - \lfloor \beta^k/d^{k+1} \rfloor - \gamma^k/d^{k+1}$$

and hence,

$$\lceil a^{k+1}/d^{k+1}\rceil = \lceil a^k/d^k\rceil - \lfloor \beta^k/d^{k+1}\rfloor + \lceil -\gamma^k/d^{k+1}\rceil.$$

But since $d^{k+1} > \gamma^k \geqslant 0$, we have $0 \geqslant -\gamma^k/d^{k+1} > -1$, and thus $\lceil -\gamma^k/d^{k+1}\rceil = 0$. This completes the proof. □

Lemma 2. $\gamma^k = \beta^{k+1}$.

Proof. The expression for $\lceil a^{k+1}/d^{k+1}\rceil$ in Lemma 1 and the definition of β^{k+1} give

$$\beta^{k+1} = (\lceil a^k/d^k\rceil - \lfloor \beta^k/d^{k+1}\rfloor)d^{k+1} - a^{k+1}.$$

Moreover, since $d^{k+1} = d^k - 1$ and $a^{k+1} = a^k - \lceil a^k/d^k\rceil$, the foregoing is equal to

$$\lceil a^k/d^k\rceil d^k - a^k - \lfloor \beta^k/d^{k+1}\rfloor d^{k+1} = \beta^k - \lfloor \beta^k/d^{k+1}\rfloor d^{k+1}.$$

But this last expression is just the definition of γ^k. This completes the proof. □

Lemma 3. *If $d^{k+1} > \beta^k$, then $\lceil a^{k+1}/d^{k+1}\rceil = \lceil a^k/d^k\rceil$ and $\beta^{k+1} = \beta^k$.*

Proof. This follows directly from Lemmas 1 and 2 and the definition of γ^k, noting that $d^{k+1} > \beta^k (\geqslant 0)$ implies that $\lfloor \beta^k/d^{k+1}\rfloor = 0$. □

Lemma 4. *Let $r = \lceil d - \beta\rceil$, where $\beta \equiv \beta^0$. Then, $\beta^k = \beta$ and $\lceil a^k/d^k\rceil = \lceil a/d\rceil$ for $k \leqslant r-1$, and $a^k = a - k\lceil a/d\rceil$ for $k \leqslant r$.*

Proof. Note that $d > \beta$, and that the definition of r implies that it is the least integer such that $\beta \geqslant d^r (= d - r)$. Therefore, $r = \lceil d - \beta\rceil \geqslant 1$ and $d^{r-1} > \beta$. Thus, it follows inductively from Lemma 3 that $\beta^k = \beta$ and $\lceil a^k/d^k\rceil = \lceil a/d\rceil$ for $k = 0, 1, \ldots, r-1$, and hence, by the inductive definition of a^k, we have, $a^k = a - k\lceil a/d\rceil$ for $k = r$. □

Lemma 5. *If $0 < d^{k+1} \leqslant \beta^k$, then $\beta^{k+1} < 1$.*

Proof. From the definition of γ^k, we have $(\beta^k - \gamma^k)/d^{k+1} = \lfloor \beta^k/d^{k+1}\rfloor$, and the assumption of the lemma implies that $\lfloor \beta^k/d^{k+1}\rfloor \geqslant 1$. Thus, we obtain $\gamma^k \leqslant \beta^k - d^{k+1}$. Since $d^k > \beta^k$, we have $\beta^k - 1 < d^k - 1 = d^{k+1}$, and hence, $\beta^k - d^{k+1} < 1$. It follows then that $\gamma^k < 1$, and so by Lemma 2, we have $\beta^{k+1} < 1$. □

Lemma 6. *Let r be specified as in Lemma 4. Then, $\lceil a^k/d^k\rceil = \lceil a^r/d^r\rceil$ for k in the interval $p - 1 \leqslant k \leqslant r$.*

Proof. By the definition of r and Lemma 4, we have that $d^r = d - r \leqslant \beta = \beta^{r-1}$. Hence, by Lemma 5, we get $\beta^r < 1$. Since $d^k \geqslant 1$ for all $k \leqslant p-1$, this means that $d^k > \beta^r$ for all $k \leqslant p-1$. Hence, the proof follows from an inductive repetition of Lemma 3 for $k = r+1, \ldots, p-1$. □

Lemma 7. *If $r \leqslant p-1$, then $\lfloor \beta^{r-1}/d^r\rfloor = 1$.*

Proof. By the definition of r and Lemma 4, we know that $r \geqslant 1$, and that $\lfloor \beta^{r-1}/d^r \rfloor = \lfloor \beta/d^r \rfloor \geqslant 1$. Also, $d^{r-1} > \beta^{r-1} = \beta$, and hence, $d^r + 1 > \beta^{r-1}$. Since $r \leqslant p-1$, we have $d^r \geqslant 1$, and $2 \geqslant (d^r+1)/d^r > \beta^{r-1}/d^r$. Consequently, $\lfloor \beta^{r-1}/d^r \rfloor \leqslant 1$, establishing the desired conclusion. □

We now state the central proposition that specifies the coefficient a^p.

Proposition 1. Let $r = \lceil d - \beta \rceil$. *Then, we have*

$$a^p = \begin{cases} a - p\lceil a/d \rceil & if\ 1 \leqslant p \leqslant r, \\ a - p\lceil a/d \rceil + (p-r), & if\ \ p \geqslant r+1. \end{cases}$$

Proof. The proposition follows directly from Lemma 4 for $1 \leqslant p \leqslant r$. For $p \geqslant r+1$, and for k such that $p-1 \geqslant k \geqslant r$, we have by Lemmas 6, 1, and 4 that $\lceil a^k/d^k \rceil = \lceil a^r/d^r \rceil = \lceil a/d \rceil - \lfloor \beta^{r-1}/d^r \rfloor$, and hence by Lemma 7 that $\lceil a^k/d^k \rceil = \lceil a/d \rceil - 1$. Consequently, using this along with $\lceil a^k/d^k \rceil = \lceil a/d \rceil$ for $k \leqslant r-1$ from Lemma 4, we get inductively that

$$\begin{aligned} a^p &= a - \sum_{t=1}^{p} \lceil a^{p-t}/d^{p-t} \rceil \\ &= a - \sum_{t=1}^{p-r} \lceil a^{p-t}/d^{p-t} \rceil - \sum_{t=p-r+1}^{p} \lceil a^{p-t}/d^{p-t} \rceil \\ &= a - (p-r)(\lceil a/d \rceil - 1) - r\lceil a/d \rceil \\ &= a - p\lceil a/d \rceil + (p-r). \end{aligned}$$

This completes the proof. □

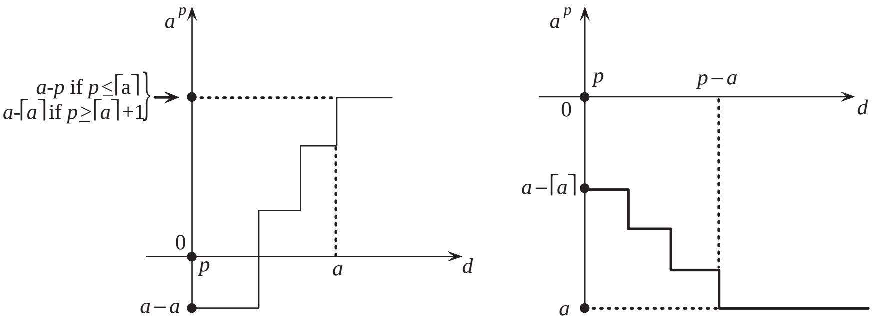
To summarize, we have shown that the source row (1) yields the cut (7), where each a_i^p, $i = 0, 1, \ldots, n$, is as given by Proposition 1 under the assumption that p is an integer such that $1 \leqslant p \leqslant d$. Accordingly, we can write (7) more explicitly as follows, where z is also a nonnegative integer variable:

$$z + \sum_{i=1}^{n} [a_i - p\lceil a_i/d \rceil + (p - r_i)^+]x_i = a_0 - p\lceil a_0/d \rceil + (p - r_0)^+, \tag{7$'$}$$

where $(\alpha)^+ \equiv \max\{0,\ \alpha\}$, and where $r_i \equiv \lceil a_i + d - \lceil a_i/d \rceil d \rceil$, $\forall i = 0, 1, \ldots, n$. By using the source equation (1) in $(7)'$ and then scaling it by p, this implies the following valid inequality:

$$\sum_{i=1}^{n} (\lceil a_i/d \rceil - (1 - r_i/p)^+)x_i \geqslant \lceil a_0/d \rceil - (1 - r_0/p)^+.$$

Notice that the second coefficient terms on both sides of this inequality reflect the effect of the tier-level p, beyond the value $p = 1$ as in the ordinary CG cut (3) that is derived using the scaling parameter d.



(a) Case of $a > 0$ (b) Case of $a \leq 0$

Fig. 1. Variation in cut coefficient with $d \geqslant p$ for a fixed $p \geqslant 1$.

3. The determination of parameters

To facilitate the derivation of particular useful members of the class of CG-tier cuts, as well as to provide further insights into the structure and nature of these cuts, we study in this section certain variational properties of the cut coefficients as functions of the scaling parameter d and the tier-level parameter p. To begin with, assume that we have selected some tier-level value $p \geqslant 1$, and let us examine how the cut coefficients behave as a function of $d \geqslant p$. The following result provides some salient properties.

Proposition 2. *Let $p \geqslant 1$ be a fixed integer, and consider $d \geqslant p$. Then for any* (*variable or right-hand side*) *coefficient a, the corresponding cut coefficient a^p in* (7) *varies as follows as a function of d.*

(i) *If $a > 0$, then*

$$a - \lceil a \rceil \leqslant a^p \leqslant \begin{cases} a - p \ if \ \ p \leqslant \lceil a \rceil \\ a - \lceil a \rceil \ if \ \ p \geqslant \lceil +1, \end{cases} \tag{8a}$$

where a^p is a nondecreasing function of d, and where the upper bound on a^p in 8(a) *is achieved for any $d \geqslant a$.*

(ii) *If $a \leqslant 0$, then*

$$a \leqslant a^p \leqslant a - \lceil a \rceil, \tag{8b}$$

where a^p is a nonincreasing function of d, and where the lower bound on a^p is achieved for any $d \geqslant p - a$.

Proof. See the appendix.

Fig. 1 illustrates the assertion of Proposition 2, and provides the foundation for determining cuts that satisfy certain stated objectives. Given a particular value (say y) of a_0^p, the strongest cut is obtained when all the cut coefficients a_i^p on the left-hand side in (7) are at their maximum values. If $a_i > 0$ for all i, then whenever p is given, it is clear by Proposition 2 that the strongest cut will be found by determining the largest possible value of $d \geqslant p$ that will yield $a_0^p = y$. Similarly, if $a_i \leqslant 0$ for all i, then the smallest value of $d \geqslant p$ that yields $a_0^p = y$, for p given, will produce the most desirable cut. Thus, to prescribe such a suitable value for d, as well as to provide guidelines for cases where the source row might have mixed-sign coefficients, we next derive restrictions on d that will yield a given value y for any specified variable or right-hand side coefficient a^p in (7). To this end, let

$$\alpha = \lceil (a - y)/p \rceil - 1 \tag{9a}$$

and

$$\delta = p\,(\alpha + 1) - (a - y)\,. \tag{9b}$$

It may be noted that the definition of δ is of the same form as the definition of β, with $a - y$ taking the place of a, and p taking the place of d. Thus, we have that $p > \delta \geqslant 0$. In addition, $a^p = y$ implies from Proposition 1 that $a - y$ is an integer, so that δ is an integer and, therefore, $p - 1 \geqslant \delta \geqslant 0$.

Proposition 3. *Let y be given such that $a - y$ is an integer, and let $p \geqslant 1$ be an integer. Then $a^p = y$ if and only if $d \geqslant p$ and d satisfies the following additional inequalities*:

For $a > 0$:

(i) $d \geqslant a/(\alpha + 1)$,
(ii) $d < p + (y + 1)/\alpha$ *if* $\alpha \neq 0$, *and*
(iii) $d \geqslant p + y/\alpha$ *if* $\alpha \neq 0$ *and* $\delta > 0$.

For $a < 0$:

(i) $d > p + (y + 1)/\alpha$, *and*
(ii) $d \leqslant p + y/\alpha$ *if* $\delta > 0$.

Proof. See the appendix.

By Proposition 2, we see that if the bounds on d given in Proposition 3 are compatible with $d \geqslant p \geqslant 1$, then y must be selected so that $a - \lceil a \rceil \leqslant y \leqslant a - 1$ for $a > 0$, and $a \leqslant y \leqslant a - \lceil a \rceil$ for $a \leqslant 0$. Also, from Proposition 2, whenever $a > 0$, we infer an upper bound for p given by $p \leqslant a - y$, unless if $y = a - \lceil a \rceil$, whence any value of $p \geqslant \lceil a \rceil$ would also achieve this same value of $a^p = y$.

As prompted by Propositions 2 and 3, whenever a cut coefficient a^p is determined for d arbitrarily close (but not equal) to a strict upper or lower bound, it is convenient for computational purposes to know the exact manner in which this a^p will differ from the cut coefficient that is determined when d exactly equals such an unattainable bound. The next result provides the relationships that characterize this difference.

Proposition 4. *Given d^*, if $d = d^* - \varepsilon$ for $\varepsilon > 0$ and sufficiently small, we have*

(i) $\lceil a/d \rceil = \lfloor a/d^* \rfloor + 1$ *and* $r = \lfloor a - d^* \lfloor a/d^* \rfloor \rfloor + 1$ *for* $a > 0$;
(ii) $\lceil a/d \rceil = \lceil a/d^* \rceil$ *and* $r = \lceil a - d^*(\lceil a/d^* \rceil - 1) \rceil$ *for* $a \leqslant 0$.

On the other hand, if $d = d^* + \varepsilon$, then (i) *holds true for $a \leqslant 0$ and* (ii) *holds true for $a > 0$.*

Proof. These relationships may be verified from the fact that for any number z and a sufficiently small $\varepsilon > 0$, we have $\lceil z - \varepsilon \rceil = \lceil z \rceil$ and $\lceil z + \varepsilon \rceil = \lfloor z \rfloor + 1$. □

The preceding results can evidently be used to prescribe the strongest possible cut for given values of p and a_0^p provided that all $a_i > 0$ or all $a_i \leqslant 0$. In other cases, a uniformly strongest cut usually does not exist. However, when the source row has coefficients of both signs, it is quite possible to obtain stronger cuts for specified criteria than might be obtained simply by maximizing the a_i^p for $a_i > 0$ or for $a_i < 0$. For example, by examining a_i^p for a_i of one sign, we may take the corresponding maximum cut coefficient values as constraining conditions, and maximize the a_i^p for a_i of the opposite sign as a secondary objective. As before, the applicable restrictions on d may be found by Proposition 3.

It is to be noted that cuts having the structure of those associated with Gomory's Method of Integer Forms are obtained by setting $a_0^p = a_0 - \lceil a_0 \rceil$. In this instance, $d = p$ yields $a_i^p = a_i - \lceil a_i \rceil$ for all i (as in the Gomory cuts), and one obtains cuts that are always as strong or stronger than the Gomory cuts by determining the largest d (within an ε amount) such that $a_i^p = a_i - \lceil a_i \rceil$ for all $a_i \leqslant 0$ (and for $i = 0$).

So far, we have said nothing about the value to be selected for p, which has heretofore been assumed to be given. However, since $p \leqslant a - y$ is necessary to assure that $a^p = y$ when $a > 0$ and $y > a - \lceil a \rceil$, whenever such a lower bound is specified for some a_i^p such that $a_i > 0$, this relation may be used to restrict p to a suitable interval. Within this interval, there may still be a choice to be made, and we could attempt several such p-values, and select among the resulting cuts, naturally discarding uniformly dominated cuts. While this is open to computational investigations, we close this section by presenting a result that provides some useful insights into the effect on cut coefficients by simultaneously varying both p and d in a prescribed fashion.

Proposition 5. *If p and d are increased in a manner such that the increase in p is at least as much as the increase in d* (*while keeping $p \leqslant d$*), *then*

(i) *a^p will decrease or stay the same for $a > 0$, and*
(ii) *a^p will increase or stay the same for $a \leqslant 0$.*

Proof. See the appendix.

4. Numerical examples

To illustrate some of the cut possibilities that may arise, we first consider the example given by Ben-Israel and Charnes [2] to exhibit the limitation of Gomory cuts. Consider the

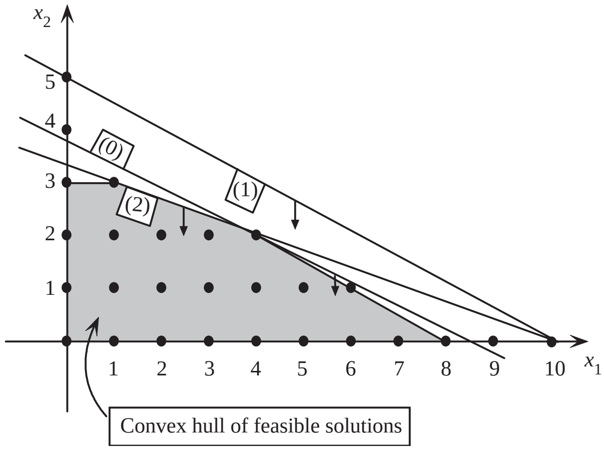


Fig. 2. Illustration for the example of Ben-Israel and Charnes [2]: (0) source row, $s + 3x_1 + 7x_2 = 26$; (1) CG cut, $z + x_1 + 2x_2 = 10$ via $p = 1$, $d < 26/15$; (2) CG-tier cut, $z + x_1 + 3x_2 = 10$ via $p = 2$, $d < 25/7$.

source row

$$s + 3x_1 + 7x_2 = 26, \tag{10}$$

where s, x_1, and x_2 are nonnegative integer variables, and s may be viewed as a slack variable for the inequality $3x_1 + 7x_2 \leqslant 26$. The equation

$$z + x_1 + 3x_2 = 10 \tag{11}$$

with z being a nonnegative integer, is a facet of the convex hull of feasible (integer) solutions to the given source row in the (x_1, x_2)-space as depicted in Fig. 2, but there is no value of the parameter λ, with $\lambda > 0$, in the standard Gomory all-integer cut

$$z + \lfloor 1/\lambda \rfloor s + \lfloor 3/\lambda \rfloor x_1 + \lfloor 7/\lambda \rfloor x_2 = \lfloor 26/\lambda \rfloor \tag{12}$$

that will yield (or imply) Eq. (11).

On the other hand, the family of CG-tier cuts, when applied to the source row (10), indeed recovers the cut (11). To see this note that we wish to obtain the strongest cut such that $a_0^p = 10$ and such that the coefficients of x_1 and x_2 are positive. Thus, since the coefficient of x_1 is 3 in the source row, and by Proposition 2, we have $a_1^p \leqslant 3 - p$ if $p \leqslant 3$, and $a_1^p = 0$ if $p \geqslant 4$, we are restricted to use $p = 1$ or $p = 2$.

The CG-tier cut (7) for $p = 1$ corresponds directly to the Gomory all-integer cut, given any selected scaling parameter value $d \geqslant 1$. We examine this case to verify the observation by Ben-Israel and Charnes that no Gomory cut (i.e., no choice of $d \geqslant 1$) will give the desired cut equation (11). With $p = 1$, we get that $y = a_0^1 = 10$ in (9a,b) implies that $\alpha = 15$, $\delta = 0$, and so, by Proposition 3, we must restrict d according to $\frac{26}{16} \leqslant d < \frac{26}{15}$. By Proposition 2, then, the strongest cut with $a_0^1 = 10$ is obtained for $d = \frac{26}{15} - \varepsilon$, for $\varepsilon > 0$ and sufficiently small. Using this value of d in Proposition 1 and using Proposition 4 to derive the corresponding cut coefficients for x_1 and x_2, we get the cut $z + x_1 + 2x_2 = 10$. (This same cut is obtained as the strengthened mixed-integer Gomory cut (see [24]) in this case, for the given value of d.) This is depicted in Fig. 2, and is in fact, dominated by the source row itself over the nonnegative quadrant.

However, when $p = 2$, $a_0^2 = 10$ implies via Proposition 3 that $26/8 \leqslant d < 25/7$, and by taking d arbitrarily close to its upper bound, we indeed obtain via Proposition 1 the corresponding CG-tier cut as $z + x_1 + 3x_2 = 10$. This cut defines a facet of the convex hull of integer feasible solutions to the source row as seen from Fig. 2.

It is interesting to compare the foregoing facetial cut with the Gomory fractional cut derived from the source row (1) after scaling this row by dividing by a similar value of d (without requiring that $a_0^1 = 10$), which is equivalent to (3) and is given by

$$\sum_i (\lceil a_i/d \rceil - a_i/d)x_i \geqslant \lceil a_0/d \rceil - a_0/d, \tag{13}$$

as well as with the corresponding strengthened Gomory mixed-integer cut (see [24]):

$$\sum_{i:f_i<f_0} \left(\frac{1-f_0}{f_0}\right) f_i x_i + \sum_{i:f_i \geqslant f_0} (1-f_i)x_i \geqslant (1-f_0), \tag{14}$$

where $f_i \equiv a_i/d - \lfloor a_i/d \rfloor$, $\forall i = 0, 1, \ldots, n$. From above, based on Proposition 3, let us take $d = 3.5 \in [26/8, 25/7)$, for which it can be verified that (11) is obtained as the CG-tier cut with $p = 2$. For this same scaling parameter d, the cuts (13) and (14) are, respectively, given by

$$\left(\tfrac{5}{7}\right) s + \left(\tfrac{1}{7}\right) x_1 \geqslant \tfrac{4}{7} \tag{15a}$$

and

$$\left(\tfrac{8}{21}\right) s + \left(\tfrac{1}{7}\right) x_1 \geqslant \tfrac{4}{7}. \tag{15b}$$

Note that in the (s, x_1, x_2)-space, (15b) uniformly dominates (15a). Translating these cuts into the (x_1, x_2)-space by substituting $s = 26 - 3x_1 - 7x_2$ from the source row, cuts (15a) and (15b), respectively, become

$$2x_1 + 5x_2 \leqslant 18 \tag{16a}$$

and

$$3x_1 + 8x_2 \leqslant 28. \tag{16b}$$

While there is no uniform dominance exhibited between these cuts and (11) over the nonnegative quadrant, observe that unlike (11), both cuts (16a) and (16b) are nonfacetial, although they both support the convex hull of feasible solutions at the point $(x_1, x_2) = (4, 2)$. We also experimented with using different values of $d = 182/(7m + 3)$, with $m \geqslant 0$ and integer-valued, for which $(1 - f_0)$ in (14) would equal 4/7 as in (15b). (Note that (15b) corresponds to $m = 7$ in this context.) By examining the resulting cuts (14) in the (x_1, x_2)-space in the scaled form $\xi_1 x_1 + \xi_2 x_2 \leqslant 10$ in order to compare against (11), we discovered that the strongest cut with $\xi_1 = 1$ was produced for values of $m = 10, 36, 62, \ldots,$ for which the cut

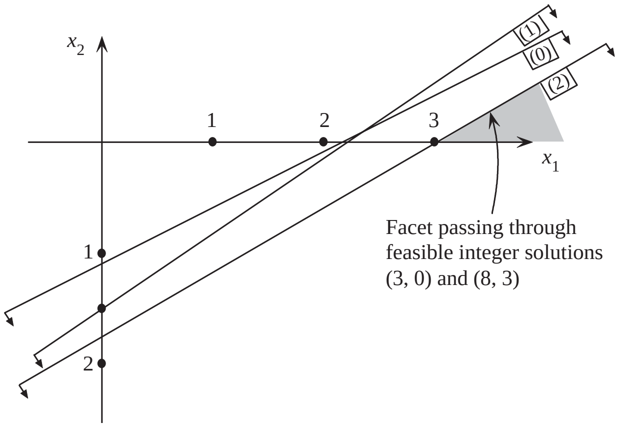


Fig. 3. Illustration of example that focuses on maximizing negative cut coefficients: (0) source row, $s - 11x_1 + 21x_2 = -25$; (1) CG cut, $z - 4x_1 + 6x_2 = -9$ via $p = 1$, $d > 25/17$; (2) CG-tier cut, $z - 3x_1 + 5x_2 = -9$ via $p = 8$, $d > 32/3$.

obtained is given by $x_1 + (277/104)x_2 \leqslant 10$. Observe that the coefficient $\xi_2 = 2.6634615$ of x_2 in this inequality in less than 3 in comparison with (11), and that this cut does not support the convex hull of feasible solutions.

In Fig. 3 and 4, we have graphed other source rows and related CG-tier cuts to illustrate certain additional circumstances in which CG-tier cuts for $p \geqslant 2$ may envelope the convex hull of feasible integer solutions more effectively than the cuts for $p = 1$. In each of these figures, given the specified source row, the cut $p = 1$ represents the CG cut, and the cut for the specified $p \geqslant 2$ (prescribed arbitrarily in these cases) is a higher level CG-tier cut. For each specified *p*-value and right-hand-side coefficient in these cases, the prescribed value for *d* is determined via Propositions 2 and 3 as the best value according to the selected cut-coefficient optimization criterion. In particular, we have selected the value of *d* to maximize a_i^p for $a_i < 0$ in Fig. 3, and to maximize a_i^p for $a_i > 0$ in Fig. 4. Note that even for the CG cut for $p = 1$, we have selected the scaling parameter *d* according to these results so as to optimize the resulting cut coefficients as indicated. In Fig. 4, where the coefficients of the source row are not integers, and we wish to obtain cuts having the structure associated with the Method of Integer Forms, then we also display the CG cut obtained by setting $p = d = 1$, in addition to deriving the best CG-tier cut for $p = 1$, which is the best cut corresponding to a value of *d* selected via Propositions 2 and 3 so as to optimize the specified (positive) cut coefficients.

Fig. 3 illustrates that a CG-tier cut with $p \geqslant 2$ may have a positive coefficient smaller than in the Gomory cut, and at the same time, possess a negative coefficient larger than in the Gomory cut. Of course, the reverse can occur as well. However, this “counterbalancing” of positive and negative coefficients (when it occurs) does not necessarily imply that the CG-tier cut for $p \geqslant 2$ is stronger than the Gomory cut over one portion of the feasible space and weaker over another. In Fig. 3, for example, the CG-tier cut for $p = 8$ defines a facet of the convex hull of integer solutions feasible to the source equation and strictly dominates

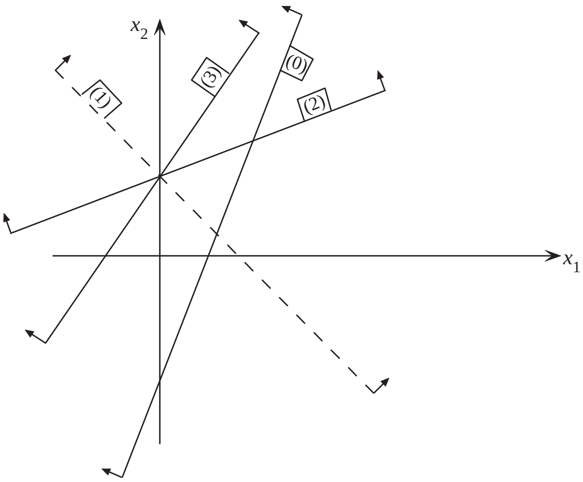


Fig. 4. Example of CG cut with and without scaling versus CG-tier cuts while maximizing positive cut coefficients: (0) source row, $s + 31.2x_1 - 7.6x_2 = 13.4$; (1) CG cut via $p = d = 1$, $z - 0.8x_1 - 0.6x_2 = -0.6$; (2) CG cut via $p = 1$, $d < 134/30$: $z + 0.2x_1 - 0.6x_2 = -0.6$; (3) CG-tier cut via $p = 7$, $d < 37/5$: $z + 1.2x_1 - 0.6x_2 = -0.6$.

the Gomory cut (which in this instance, is also dominated by the source row) over the nonnegative quadrant. While there are Gomory cuts for other values of a_0^p that do better than the one illustrated here, there are none that replicate the facetial CG-tier cut depicted here.

Fig. 4 further illustrates a situation that may arise when dealing with cuts susceptible to exploitation by Gomory's Method of Integer Forms. Here, the Gomory cut excludes a portion of the continuous feasible space defined by the source row, but is strictly dominated in the nonnegative quadrant by the CG-tier cut for $p = 1$ that selects d to maximize a_i^p for $a_i > 0$. At the same time, the latter cut is itself strictly dominated in the nonnegative quadrant by the CG-tier cut for $p = 7$, which, in fact, happens to support the convex hull of integer feasible solutions. The reader might observe in Fig. 4 that the depicted cuts exclude certain integer points in the (x_1, x_2)-space that appear to be feasible to the source constraint, as for example, the origin. This is not contrary to the validity of these cuts due to the fact that the s variable in the source equation is also restricted to be a nonnegative *integer*. Thus, the admissible integer points must satisfy $0.2x_1 - 0.6x_2 = 0.4 \pmod 1$. Hence, in particular, the CG-tier cut with $p = 7$ shown in Fig. 4 supports the convex hull of the integer feasible solutions at this point $(x_1, x_2, s) = (1, 3, 5)$.

5. Summary and conclusions

With the advent of more sophisticated LP solvers and branch-and-cut implementations, accompanied by a revolution in computer technology, there has been a recent resurgence of interest in Chvatal–Gomory (CG) cuts. In the present paper, we have introduced a new class of Chvatal–Gomory (CG)-tier cuts that employs a scaling parameter d on a suitable source row to generate a CG cut at level one, and then progressively repeats this process up to a tier-level p, $1 \leqslant p \leqslant d$, each time using the previous cut as a source row along with a scaling parameter that is progressively decremented by one.

We have shown how such a cut can be immediately written in closed-form, predicated on the choice of the respective scaling and tier-level parameters d and p. Moreover, we have provided a detailed analysis to study the variation in the cut coefficients as a function of varying $d \geqslant p$ for a fixed $p \geqslant 1$, and also, varying both p and d simultaneously in a specific manner. This has enabled us to prescribe choices of d and p that would generate strong CG-tier cuts from the viewpoint of tightening cut coefficients along specified desired dimensions. Several examples have been presented to provide insights into the nature and efficacy of CG-tier cuts, including the potential of deriving strong cuts that might not be possible to obtain by employing a rank-one CG cut.

The focus in this paper has been to introduce this class of CG-tier cuts and to establish the theoretical underpinnings of its structure as determined by its driving parameters. A follow-on study will consider computational implementation aspects, including a detailed investigation of formulating and solving related separation problems that involve the derivation of a suitable source row via a surrogate analysis, and the subsequent generation of a CG-tier cut that is strong with respect to a specified cut objective. Moreover, we could explore the generation of rounds of CG-tier cuts as in Balas et al. [1], both from the same as well as from alternative source rows. For further research, it might also be insightful to explore special cases when particular CG-tier cuts might yield facet-defining inequalities for the closure convex hull of integer feasible solutions, as well as to compare CG-tier cuts with other classes of related cuts such as Gomory's mixed-integer (or strengthened fractional) cuts. In addition, it is of interest to conduct a similar analysis as in this paper to generate CG-tier cuts for mixed-integer programs. This could be done by either generating the CG-tier cuts proposed herein directly on original or implied source rows that involve only integer variables, or to establish a recursive Gomory mixed-integer cut process (see [24]) via successive disjunctions on multiple fractionating integer variables that are included within a suitable source row. This investigation, along with related computational experiments, is recommended for future research.

Acknowledgements

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Appendix

In order to prove Proposition 2, consider first the following technical results.

Lemma A.1. $\lceil a/d \rceil = \lceil (a - a^p)/p \rceil$.

Proof. Since $r \geqslant 1$, we have from Proposition 1 that

$$a - p\lceil a/d\rceil \leqslant a^p \leqslant a - p\lceil a/d\rceil + p - 1$$

and hence, $p\lceil a/d\rceil \geqslant a - a^p \geqslant p\lceil a/d\rceil - p + 1$. Dividing through by p, and observing that $1/p - 1 > -1$, we obtain $\lceil a/d\rceil \geqslant (a - a^p)/p > \lceil a/d\rceil - 1$. The lemma follows directly from these inequalities. □

Lemma A.2. *Let $p \geqslant 1$ be a constant integer, and let d be constrained so that $d \geqslant p$. Then, as d increases*

(i) *a^p increases or stays the same for $a > 0$, and*
(ii) *a^p decreases or stays the same for $a \leqslant 0$.*

Proof. First, assume that as d increases, $\lceil a/d\rceil$ stays constant. We note that $\lceil a/d\rceil \geqslant 1$ for $a > 0$, and $\lceil a/d\rceil \leqslant 0$ for $a \leqslant 0$. Since $r = \lceil d - \beta\rceil = \lceil a - d(\lceil a/d\rceil - 1)\rceil$, it follows that d must have the effect stated in the lemma in this case. Next, suppose that as d increases, the value of $\lceil a/d\rceil$ changes. Then, $\lceil a/d\rceil$ must decrease for $a > 0$ and increase for $a < 0$. But by Lemma A.1, with p constant, this can only happen if a^p increases in the first instance and decreases in the second, thus completing the proof. □

Lemma A.3. *$a - \lceil a\rceil$ is the minimum value of a^p when $a > 0$ and the maximum value of a^p when $a \leqslant 0$. Moreover, $a^p = a - \lceil a\rceil$ whenever $d = p$.*

Proof. For any *given* value of p, the minimum value of a^p for $a > 0$ and the maximum value of a^p for $a \leqslant 0$ occur by Lemma A.2 whenever d is selected so that $d = p$. In this event, $r = \lceil a - p(\lceil a/p\rceil - 1)\rceil$, and since p is a positive integer, $r = \lceil a\rceil - p\lceil a/p\rceil + p$. Thus, $p - r = p\lceil a/p\rceil - \lceil a\rceil$. This latter quantity is nonnegative since $p\lceil a/p\rceil - a \geqslant 0$ and $p\lceil a/p\rceil$ is an integer. Thus, by Proposition 1, whether $p = r$ or $p \geqslant r + 1$, we have $a^p = a - p\lceil a/p\rceil + p\lceil a/p\rceil - \lceil a\rceil = a - \lceil a\rceil$ for $p = d$. This completes the proof. □

Lemma A.4. *The minimum value of a^p when $a \leqslant 0$ is a, and is obtained for any d satisfying $d \geqslant p - a$. The maximum value of a^p when $a > 0$ is $a - p$ if $p \leqslant \lceil a\rceil$, and is $a - \lceil a\rceil$ if $p \geqslant \lceil a\rceil + 1$, where this maximum is achieved for any $d \geqslant a$.*

Proof. First, suppose that $a \leqslant 0$. For $d \geqslant p - a$, we have $1 \geqslant p/d - a/d$, and hence, $a/d > -1$. Thus, $\lceil a/d\rceil = 0$ and $r = \lceil a + d\rceil \geqslant p$. Therefore, $p - r \geqslant 0$, and from Proposition 1, $a^p = a - p\lceil a/d\rceil = a$. The fact that a^p is independent of d, for all $d \geqslant p - a$, assures from Lemma A.2 that a^p is at a minimum.

Next, suppose that $a > 0$. For $d \geqslant a$, it may be verified that $r = \lceil a\rceil$, from which the second part of the lemma follows in a manner analogous to the first. □

Proof of Proposition 2. The result now follows directly from Lemmas A.2–A.4.

Proof of Proposition 3. For a^p to be well-defined it is, of course, necessary that $d \geqslant p$, and we take this to be implicit throughout the following. To obtain a value of d that will yield $a^p = y$, this d must assure that $\lceil a/d \rceil = \alpha + 1$ by Lemma A.1. This will occur iff (if and only if) $\alpha + 1 \geqslant a/d > \alpha$, or

$$d(\alpha + 1) \geqslant a > d\alpha. \tag{17}$$

Moreover, assuming that a d exists to satisfy (17), we have by the definition of δ that $y = a - p\lceil a/d \rceil + \delta$. Thus, $a^p = y$ iff (17) is satisfied and

$$p - r = \delta \quad \text{for} \quad \delta > 0,$$

$$p - r \leqslant \delta \quad \text{for} \quad \delta = 0.$$

In both cases, we must have $p - r \leqslant \delta$. Since $r = \lceil d - \beta \rceil$, we have that $p - r \leqslant \delta$ iff $p = -\delta < d - \beta + 1$. But from the definition of β, we have $d - \beta = a - d\alpha$, yielding

$$d\alpha < a + \delta - p + 1. \tag{18}$$

Thus, (17) and (18) supply necessary and sufficient restrictions on d to yield $a^p = y$ if $\delta = 0$. To accomplish the same for $\delta > 0$, we require, in addition, $p - r \geqslant \delta$ (so that then $p - r = \delta$), which is true iff $d - \beta \leqslant p - \delta$. Following the argument that gave (18), we find this last inequality equivalent to

$$d\alpha \geqslant a + \delta - p. \tag{19}$$

Moreover, the upper and lower bounds on $d\alpha$ given by (18) and (19), respectively, are at least as restrictive as those of (17). Comparing (18) to (17), this follows from $\delta \leqslant p - 1$. Comparing (19)–(17), this again follows provided that we have $d + \delta - p \geqslant 0$. But this is immediate from $d \geqslant p$ and $\delta \geqslant o$ (here actually, $\delta \geqslant 1$).

Replacing δ in (18) and (19) by its definition, we obtain

$$d\alpha < p\alpha + y + 1 \text{ iff } p - r \leqslant \delta \tag{20}$$

and

$$d\alpha \geqslant p\alpha + y \text{ iff } p - r \geqslant \delta. \tag{21}$$

To summarize, $a^p = y$ iff $p\alpha + y \leqslant d\alpha < p\alpha + y + 1$ for $\delta > 0$, and $d(\alpha + 1) \geqslant a$ and $d\alpha < p\alpha + y + 1$ for $\delta = 0$. Noting that the integer $\alpha \geqslant 0$ if $a > 0$ and $\alpha \leqslant -1$ from (17), the assertions of the proposition now readily follow. $\square$

Proof of Proposition 5. For any p and d, let $p' = p + \Delta_p$ and $d' = d + \Delta_d$, where $\Delta_p \geqslant \Delta_d \geqslant 0$, and Δ_p is integral. (As always, note that $d \geqslant p$ and $d' \geqslant p'$ are assumed to hold true.) Replace y by a^p in the definition of α and δ, and let α' and δ' be defined relative to p', d' and $a^{p'}$ and, in the same way that α and δ are defined relative to p, d, and a^p. Finally, define $\Delta' = \alpha - \alpha'$. Then, from the identity for δ, we get

$$a^p = a - p(\alpha + 1) + \delta$$

and

$$a^{p'} = a - p'(\alpha' + 1) + \delta'.$$

Thus,

$$\delta' = \delta - p\Delta' + \Delta_p(\alpha' + 1) + a^{p'} - a^p.$$

We first prove (i). Hence, assume that $a > 0$, and on the contrary, suppose that $a^{p'} - a^p \geqslant 1$. By the definitions of α and α' (see also Lemma A.1), this implies that $\Delta' \geqslant 0$. Furthermore, since $\delta \geqslant 0$ and $\alpha' \geqslant 0$ (by Lemma A.1 and the definition of α'), it follows from the expression for δ' above that $\Delta' = 0$ implies that $\delta' > 0$.

Now, by (20) in the proof of Proposition 3, we have that $d\alpha < p\alpha + a^p + 1$. But $p\alpha = p'\alpha' + p'\Delta' - \alpha\Delta_p$ and $d\alpha = d'\alpha' + d'\Delta' - \alpha\Delta_d$, so that (20) yields

$$p'\alpha' + \alpha(\Delta_d - \Delta_p) - (d' - p')\Delta' + a^p + 1 > d'\alpha'. \tag{22}$$

There are two cases to consider: $\delta' > 0$ and $\delta' = 0$. First, suppose that $\delta' > 0$. Then by (21) in the proof of Proposition 3, we have that $d'\alpha' \geqslant p'\alpha' + a^{p'}$, and using this in (22) above, we get

$$\alpha(\Delta_d - \Delta_p) > (a^{p'} - a^p - 1) + (d' - p')\Delta'. \tag{23}$$

From this, the assumption $a^{p'} - a^p \geqslant 1$, along with $\Delta' \geqslant 0$ and $(d' - p') \geqslant 0$, while $\alpha \geqslant 0$ and $\Delta_d - \Delta_p \leqslant 0$, yield $0 > 0$, a contradiction. This establishes (i) for $\delta' > 0$.

Next, suppose that $\delta' = 0$. Again, from (17) in the proof of Proposition 3, we have $d'\alpha' \geqslant a - d'$. Since $\delta' = 0$, we have $a^{p'} = a - p'(\alpha' + 1)$ and hence, $a = a^{p'} + p'\alpha' + p'$. Substituting this for a in $d'\alpha' \geqslant a - d'$ and applying (22), we obtain

$$\alpha(\Delta_d - \Delta_p) > (a^{p'} - a^p - 1) + (\Delta' - 1)(d' - p'). \tag{24}$$

By our earlier remarks, $\Delta' \geqslant 1$ for $\delta' = 0$ (since $\Delta' \geqslant 0$, and $\Delta' = 0$ implies that $\delta' > 0$). Hence, similar to (23), inequality (24) again leads to a contradiction that $0 > 0$, thus completing the proof of (i).

To prove (ii), we likewise derive

$$\alpha(\Delta_p - \Delta_d) > (a^p - a^{p'} - 1) - (d' = p')\Delta' \quad \text{for} \quad \delta > 0 \tag{25}$$

and

$$(\alpha + 1)(\Delta_p - \Delta_d) > (a^p - a^{p'} - 1) - (\Delta' + 1)(d' - p') \quad \text{for} \quad \delta = 0. \tag{26}$$

Note that for the case $a \leqslant 0$, Lemma A.1 yields that $\alpha = \lceil a/d \rceil - 1 \leqslant -1$. Hence, since $\Delta_p \geqslant \Delta_d$, we have that the left-hand sides of both (25) and (26) are nonpositive. The proposition then follows again by contradiction upon assuming that $a^p - a^{p'} \geqslant 1$ and noting that this implies $\Delta' \leqslant 0$, where $\Delta' = 0$ implies that $\delta > 0$ and so, $\Delta = 0$ implies that $\Delta' \leqslant -1$. $\square$

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